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Anomalous N=2 Superconformal Ward Identities

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Abstract

The N=2 superconformal Ward identities and their anomalies are discussed in N=2 superspace (including N=2 harmonic superspace), at the level of the low-energy effective action (LEEA) in four-dimensional N=2 supersymmetric field theories. The (first) chiral N=2 supergravity compensator is related to the known N=2 anomalous Ward identity in the N=2 (abelian) vector multiplet sector. As regards the hypermultiplet LEEA given by the N=2 non-linear sigma-model (NLSM), a new anomalous N=2 superconformal Ward identity is found, whose existence is related to the (second) analytic compensator in N=2 supergravity. The celebrated solution of Seiberg and Witten is known to obey the (first) anomalous Ward identity in the Coulomb branch. We find a few solutions to the new anomalous Ward identity, after making certain assumptions about unbroken internal symmetries. Amongst the N=2 NLSM target space metrics governing the hypermultiplet LEEA are the $SU(2)$ -Yang-Mills-Higgs monopole moduli-space metrics that can be encoded in terms of the spectral curves (Riemann surfaces), similarly to the Seiberg-Witten-type solutions. After a dimensional reduction to three spacetime dimensions (3d), our results support the mirror symmetry between the Coulomb and Higgs branches in 3d, N=4 gauge theories.

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1 Introduction

Field theories with $N=2$ extended supersymmetry are known to possess remarkable properties that sometimes allow one to obtain their exact (non-perturbative) low-energy solutions, pioneered by Seiberg and Witten [1]. The natural way to produce such exact results is to relate the field theory problem to an integrable system (or Whitham dynamics in the Seiberg-Witten case) [2]. The origin of an elliptic curve behind the exact solution [1] also becomes apparent in this approach. Another (related) interpretation of the Seiberg-Witten result is possible from the viewpoint of anomalous breaking of $N=2$ superconformal symmetry and uniformization theory on Riemann surfaces [3]. It is not very surprising since $N=2$ superconformal symmetry is well-known to be instrumental in deriving the classical structure of all (non-conformal) $N=2$ supersymmetric field theories, including $N=2$ supergravity, in four spacetime dimensions. It is natural to examine first the constraints imposed by $N=2$ superconformal invariance and then study compensation of unwanted $N=2$ superconformal symmetries. Or one can start from a classical field theory that is $N=2$ superconformally invariant, and then investigate its superconformal anomalies that can be developed in quantum field theory.

Much work in the recent past was devoted to investigating the constraints imposed by $N=4$ superconformal symmetry on the correlation functions of the $N=4$ super-Yang-Mills theory in the context of AdS/CFT correspondence [4]. More recently, similar constraints on the correlation functions were studied in the context of four-dimensional $N=2$ conformal supersymmetry [5]. We recall that all $N=2$ supersymmetric field theories can be brought into the manifestly $N=2$ supersymmetric form, by using *off-shell* (unconstrained) $N=2$ superfields in *Harmonic Superspace* (HSS) [6]. It makes the effective field theory methods to be very efficient in the $N=2$ case contrary to the $N=4$ case where only on-shell $N=4$ supersymmetry is possible. The next obvious step is to make manifest $N=2$ superconformal invariance of the $N=2$ supersymmetric quantum effective action, and then study its superconformal anomalies. Truly non-perturbative results are expected to be obtained along these lines.

As regards quantum field theories with rigid $N=2$ supersymmetry, their building blocks are given by $N=2$ vector multiplets and hypermultiplets. At the level of the *Low-Energy Effective Action* (LEEA), on the (abelian) $N=2$ vector multiplet side we have to deal with the Seiberg-Witten-type action specified by a holomorphic potential and the associated special Kähler geometry. This topic is well-known, and we are going to use it as our basic pattern to follow. Hypermultiplets (e.g., if they are

magnetically charged) may also develop the non-trivial LEEA that takes the form of a hyper-Kähler *Non-Linear Sigma-Model* (NLSM) by N=2 supersymmetry. After being formulated in HSS, the N=2 NLSM also possess an (analytic) potential. Our purpose in this paper is to impose and make manifest N=2 superconformal invariance in the N=2 NLSM, and then formulate an anomalous N=2 superconformal Ward identity on the hyper-Kähler potential.

The paper is organized as follows. In sect. 2 we briefly review the N=2 supercurrents, the anomalous N=2 superconformal Ward identities in the N=2 gauge sector, and their relation to the solution of Seiberg and Witten [1]. In sect. 3 we construct in HSS the most general N=2 superconformal NLSM that gives a general solution to the special hyper-Kähler geometry. The anomalous N=2 superconformal Ward identity on the hypermultiplet LEEA (N=2 NLSM) is found for the first time in sect. 4, by using the N=2 supergravity compensators in HSS. In sect. 5 we give our conclusions. In Appendix we briefly review a construction of N=2 supergravity in HSS.

2 N=2 supercurrent and Ward identities

An N=2 supercurrent is the irreducible representation of N=2 supersymmetry in four spacetime dimensions, having superspin one. The independent field components of the N=2 supercurrent (of some N=2 matter system) include the energy-momentum tensor, the N=2 supersymmetry current, the central charge current, the axial current, the $SU(2)$ current of R-symmetry and some auxiliary field components of lower dimension.

The relevant field components of the N=2 supercurrent were first identified in ref. [7] by analyzing the free field theory of a massive (Fayet-Sohnius) hypermultiplet. The systematic way of derivation of the N=2 supercurrent superfield is provided by a construction of the irreducible N=2 superfields in the conventional (flat) N=2 superspace $\{\mathcal{Z}\} = (x^m, \theta_\alpha^i, \bar{\theta}_{\dot{j}}^\alpha)$, where $m = 0, 1, 2, 3$, $\alpha = 1, 2$ and $i, j = 1, 2$. All irreducible N=2 superprojectors were found in ref. [8], whereas all irreducible (scalar) N=2 superfields were explicitly derived in ref. [9]. Amongst the irreducible N=2 superfields, comprising a general N=2 real scalar superfield, one finds almost all off-shell N=2 supermultiplets that usually appear in any discussion of N=2 supersymmetry (with a finite number of the auxiliary fields). In particular, an N=2 (restricted) chiral superfield $\Phi(x, \theta, \bar{\theta})$ is defined by the N=2 superspace (off-shell) constraints

$$\bar{D}_{\dot{\alpha}}^i \Phi = 0, \quad D^4 \Phi = \square \bar{\Phi}, \quad (2.1)$$

where $(D_i^\alpha, \bar{D}_{\dot{\alpha}}^j)$ are the usual (flat) N=2 superspace covariant derivatives.² As a consequence of the constraints (2.1), the N=2 restricted chiral superfield Φ possess a two-form $F = F_{mn}dx^m \wedge dx^n$ satisfying the ‘Bianchi identity’ $dF = 0$. A solution to the ‘Bianchi identity’, $F = dA$, in terms of the one-form A subject to the gauge transformations $\delta A = d\lambda$, allows one to represent the N=2 superfield strength W of an abelian N=2 vector multiplet by a restricted chiral N=2 superfield too.

The N=2 restricted chiral superfield Φ is dual to an N=2 linear superfield L^{ij} . The latter is symmetric with respect to its $SU(2)$ indices and satisfies the off-shell constraints

$$D_{\alpha}^{(i} L^{jk)} = \bar{D}_{\dot{\alpha}}^{(i} L^{jk)} = 0, \quad \overline{(L^{ij})} = \varepsilon_{ik}\varepsilon_{jl}L^{kl}. \quad (2.2)$$

The duality relation in N=2 superspace is just given by

$$L^{ij} = D^{ij}\Phi. \quad (2.3)$$

Both superfields Φ and L^{ij} represent the irreducible N=2 multiplets of superspin zero and superisospin zero.

Similarly, an irreducible N=2 scalar superfield R of superspin zero and superisospin one is defined by the constraints [9]

$$D_{\alpha\beta}R = \bar{D}_{\dot{\alpha}\dot{\beta}}R = i[D_{\alpha}^j, \bar{D}_{j\dot{\alpha}}]R = 0. \quad (2.4)$$

The irreducible N=2 scalar superfield R is dual to an N=2 projective superfield T^{ijkl} ,

$$T^{ijkl} = D^{(ij}\bar{D}^{kl)}R, \quad (2.5)$$

where T^{ijkl} is totally symmetric with respect to its $SU(2)$ indices and obeys the (off-shell) constraints

$$D_{\alpha}^{(i} T^{jklm)} = \bar{D}_{\dot{\alpha}}^{(i} T^{jklm)} = 0, \quad \overline{(T^{i_1i_2i_3i_4})} = \varepsilon_{i_1j_1}\varepsilon_{i_2j_2}\varepsilon_{i_3j_3}\varepsilon_{i_4j_4}T^{j_1j_2j_3j_4}. \quad (2.6)$$

The N=2 supercurrent J is also in the list of the irreducible N=2 scalar superfields, being defined by the constraints [9]

$$D_{ij}J = \bar{D}_{\dot{i}\dot{j}}J = 0. \quad (2.7)$$

The N=2 superspace constraints (2.7) imply that the energy-momentum tensor is symmetric, conserved and traceless, whereas all the vector currents are conserved. In other words, J is a multiplet of N=2 superconformal currents.

²We use the notation $D_{\alpha\beta} = D_{i\alpha}D_{\beta}^i$, $D^{ij} = D^{i\alpha}D_{\alpha}^j$ and $D^4 = \frac{1}{12}D_{ij}D^{ij}$, and similarly for the conjugated quantities.

An N=2 superconformal anomaly amounts to breaking the N=2 supercurrent conservation relations (2.7) by an N=2 anomaly multiplet of lower superspin, i.e. of superspin zero. This is equivalent to activating an irreducible superspin-zero superfield in the N=2 scalar superfield J . As is clear from the above discussion, there are potentially *two* ways of assigning the N=2 anomaly multiplet with an N=2 irreducible superspin-zero superfield: either Φ (or, equivalently, L^{ij}) or R (or, equivalently, L^{ijkl}). The main difference between the two choices is the fact that L^{ij} still contains a conserved vector current (associated with unbroken central charge transformations), whereas the vector current in L^{ijkl} is not conserved. The first choice yields the N=2 superconformal anomaly relation in the standard form [10]

$$\frac{i}{4}D_{ij}J = L_{ij} . \quad (2.8)$$

For example, the N=2 supercurrent conservation law in the quantum N=2 SYM theory takes the form [10, 11]

$$\frac{i}{4}D_{ij}J = \bar{D}_{ij}\bar{S} , \quad S = \frac{c}{2}\text{tr}W^2 , \quad (2.9)$$

where the (Lie algebra-valued) N=2 SYM superfield strength W has been introduced. The constant c is proportional to the one-loop renormalization group beta-function. Though $\text{tr}W^2$ is merely a chiral (not a restricted chiral) N=2 superfield, eq. (2.9) can be easily brought into the form (2.8) by a local shift of the supercurrent, $J \rightarrow J - 4iS$.

The most general N=2 supersymmetric *Ansatz* for the LEEA of some number (r) of abelian N=2 vector multiplets is governed by a holomorphic potential $\mathcal{F}(W)$ of the N=2 (restricted chiral) superfield strengths W_p [12, 13],³

$$I[W] = \int d^4x d^4\theta \mathcal{F}(W_p) + \text{h.c.} \quad (2.10)$$

The superfield W has conformal weight +1, in accordance with its canonical dimension, whereas the N=2 chiral superspace measure in eq. (2.10) has conformal weight (-2). It is, therefore, clear that the N=2 superconformal Ward identity for the LEEA (2.10) is given by

$$\sum_{p=1}^r W_p \frac{\partial \mathcal{F}}{\partial W_p} - 2\mathcal{F} = 0 . \quad (2.11)$$

The N=2 superconformal solution to the holomorphic potential $\mathcal{F}(W)$ is thus given by a homogeneous (of degree two) function. The (rigid) N=2 superconformal invariance of the action (2.10) is the necessary pre-requisite for its coupling to N=2 conformal supergravity [12].

³Our normalization differs from that used in ref. [1] by a factor $-i/(16\pi)$.

Given a non-trivial renormalization flow (like in the N=2 SYM theory), the N=2 superconformal Ward identity (2.11) is going to be broken by the anomaly S ,

$$\sum_{p=1}^r W_p \frac{\partial \mathcal{F}}{\partial W_p} - 2\mathcal{F} = 4S . \quad (2.12)$$

This equation is just the anomalous N=2 superconformal Ward identity for (abelian) N=2 vector multiplets, which was found in ref. [14] by ‘averaging’ the anomaly relation (2.9) with respect to the quantum effective action (2.10). A simple derivation of eq. (2.12) by using the N=2 supergravity compensators is discussed in sect. 4.

Equation (2.12) can be applied to a derivation of the Seiberg-Witten solution [1] provided that one knows the anomaly S as the function of W , in the context of the N=2 SYM theory based on the gauge group $SU(2)$ spontaneously broken to $U(1)$. The original derivation [1] made use of the electric-magnetic duality and renormalization flow. Since the anomaly S is N=2 chiral, gauge-invariant and of dimension two, its vacuum expectation value has to be proportional to the order parameter $u = \frac{1}{2} \langle \text{tr} W^2 \rangle$, with the coefficient being dictated by the one-loop beta-function β_1 — see eq. (2.9). A comparison with ref. [1] yields [14]

$$c = 2\pi i \beta_1 . \quad (2.13)$$

To close eq. (2.12), as the equation on \mathcal{F} , one needs a relation between $a = \langle W \rangle$ and u . It was obtained in ref. [3] by using the modular invariance of $u = u(a)$, in the form of a non-linear differential equation,

$$(1 - u^2)u'' + \frac{1}{4}au'^3 = 0 . \quad (2.14)$$

A connection to integrable systems arises after identifying the moduli space of the Coulomb branch in the Seiberg-Witten model with the moduli space of complex structures on an elliptic curve. The Seiberg-Witten solution then appears to be a classical solution to the equations of motion of a particular spin chain system [2], while the origin of the elliptic curve underlying the dynamics becomes apparent in this approach.⁴ Generalizations to the larger gauge groups and the presence of N=2 matter are straightforward, in principle.

Our main goal, however, is to explore what can happen on the hypermultiplet side, at the level of the LEEA. Unlike the N=2 vector multiplets, the universal and most symmetric off-shell formulation of hypermultiplets is only possible in HSS, with the infinite number of the auxiliary fields.

⁴The origin of the elliptic curve in the Seiberg-Witten exact solution is also explained by brane technology in the context of M-theory [15].

3 Rigid N=2 superconformal symmetry in HSS

In the HSS approach [6] the standard N=2 superspace coordinates $\{\mathcal{Z}\} = (x^m, \theta_\alpha^i, \bar{\theta}_{\dot{\alpha}}^j)$ are extended by bosonic harmonics (or twistors) $u^{\pm i}$, $i = 1, 2$, belonging to the group $SU(2)$ and satisfying the unimodularity condition

$$u^{+i}u_i^- = 1, \quad \overline{u^{i+}} = u_i^- . \quad (3.1)$$

The hidden analyticity structure of the N=2 superspace constraints defining both N=2 vector multiplets and FS hypermultiplets, as well as their solutions in terms of unconstrained N=2 superfields, can be made manifest in HSS [16].

Instead of an explicit parametrization of the twistor sphere $S^2 = SU(2)/U(1)$, the $SU(2)$ -covariant HSS approach deals with the equivariant functions of harmonics, having the definite $U(1)$ charges defined by $U(u_i^\pm) = \pm 1$. The simple harmonic integration rules,

$$\int du = 1 \quad \text{and} \quad \int du u^{+(i_1} \dots u^{+i_m} u^{-j_1} \dots u^{-j_n)} = 0 \quad \text{otherwise} , \quad (3.2)$$

are similar to the (Berezin) integration rules in superspace. In particular, any harmonic integral over a $U(1)$ -charged quantity vanishes. The harmonic covariant derivatives, preserving the defining equations (3.1) in the original (central) basis, are given by

$$\partial^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} , \quad \partial^{--} = u^{-i} \frac{\partial}{\partial u^{+i}} , \quad \partial^0 = u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} . \quad (3.3)$$

They satisfy an $su(2)$ algebra and commute with the standard (flat) N=2 superspace covariant derivatives D_i^α and $\bar{D}_{\dot{\alpha}}^j$. The operator ∂^0 measures $U(1)$ charges.

The key feature of HSS is the existence of the *analytic* subspace parametrized by

$$(\zeta^M; u) = \left\{ x_{\text{analytic}}^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} - 4i\theta^{i\alpha}\bar{\theta}^{\dot{\alpha}j}u_{(i}^+u_{j)}^-, \quad \theta_\alpha^+ = \theta_\alpha^i u_i^+, \quad \bar{\theta}_{\dot{\alpha}}^+ = \bar{\theta}_{\dot{\alpha}}^i u_i^+; \quad u_i^\pm \right\} , \quad (3.4)$$

which is invariant under N=2 (rigid) supersymmetry [6]:

$$\begin{aligned} \delta x_{\text{analytic}}^{\alpha\dot{\alpha}} &= -4i \left(\varepsilon^{i\alpha} \bar{\theta}^{\dot{\alpha}+} + \theta^{\alpha+} \bar{\varepsilon}^{\dot{\alpha}i} \right) u_i^- \equiv -4i \left(\varepsilon^{\alpha-} \bar{\theta}^{\dot{\alpha}+} + \theta^{\alpha+} \bar{\varepsilon}^{\dot{\alpha}-} \right) , \\ \delta \theta_\alpha^+ &= \varepsilon_\alpha^i u_i^+ \equiv \varepsilon_\alpha^+ , \quad \delta \bar{\theta}_{\dot{\alpha}}^+ = \bar{\varepsilon}_{\dot{\alpha}}^i u_i^+ \equiv \bar{\varepsilon}_{\dot{\alpha}}^+ , \quad \delta u_i^\pm = 0 . \end{aligned} \quad (3.5)$$

The analytic dependence includes $\theta_{\hat{\alpha}}^+$ but not $\theta_{\hat{\alpha}}^-$, where $\hat{\alpha} = (\alpha, \dot{\alpha})$.

The usual complex conjugation does not preserve analyticity. However, it does, after being combined with another (star) conjugation that only acts on the $U(1)$ indices as $(u_i^+)^* = u_i^-$ and $(u_i^-)^* = -u_i^+$. One has $\overline{u^{\pm i}}^* = -u_i^\pm$ and $\overline{u_i^\pm}^* = u^{\pm i}$.

Analytic superfields $\phi^{(q)}(\zeta(\mathcal{Z}, u), u)$ of any positive (integral) $U(1)$ charge q in HSS are defined by the off-shell constraints (*cf.* the definition of N=1 chiral superfields)

$$D^+_{\alpha} \phi^{(q)} = \bar{D}^+_{\dot{\alpha}} \phi^{(q)} = 0, \quad \text{where} \quad D^+_{\alpha} = D^i_{\alpha} u_i^+ \quad \text{and} \quad \bar{D}^+_{\dot{\alpha}} = \bar{D}^i_{\dot{\alpha}} u_i^+. \quad (3.6)$$

The analytic measure reads $d\zeta^{(-4)} du \equiv d^4 x^m_{\text{analytic}} d^2 \theta^+ d^2 \bar{\theta}^+ du$, and it is of $U(1)$ charge (-4) . The covariant derivatives in the analytic basis (3.4) receive certain connection terms. For example, the harmonic derivative ∂^{++} in the analytic subspace is replaced by

$$D^{++} = \partial^{++} - 4i\theta^{\alpha+} \bar{\theta}^{\dot{\alpha}+} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}. \quad (3.7)$$

This derivative preserves analyticity and permits integration by parts. Similarly, one easily finds the $U(1)$ charge operator in the analytic subspace reads

$$D^0 = u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} + \theta^{\alpha+} \frac{\partial}{\partial \theta^{\alpha+}} + \bar{\theta}^{\dot{\alpha}+} \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}+}}. \quad (3.8)$$

In what follows we always use the analytic basis and the associated HSS covariant derivatives denoted by capital D , without making explicit references.

The use of harmonics also gives us control over the (linearly realised) $SU(2)_R$ symmetry (or its absence), in the context of manifest N=2 supersymmetry (see ref. [17] for more details). Since the translational and Lorentz symmetries, as well as N=2 supersymmetry, are manifestly realized in HSS, the latter provides us with the natural arena for a study of ‘truly’ N=2 superconformal symmetries on the top of N=2 non-conformal (rigid or Poincaré) supersymmetry.

The superfield transformation rules with respect to dilatations (with the infinitesimal parameter ρ) are dictated by conformal weights w of the superfields, together with the weights of the N=2 superspace coordinates,

$$w[x] = 1, \quad w[\theta] = w[\bar{\theta}] = \frac{1}{2}, \quad w[u] = 0. \quad (3.9)$$

The non-trivial part of the N=2 superconformal transformations is given by $SU(2)_{\text{conf}}$ internal rotations with the parameters l^{ij} , special conformal transformations with the parameters $k_{\alpha\dot{\alpha}}$, and N=2 special supersymmetry with the parameters η^i_{α} and $\bar{\eta}^{\dot{\alpha}}_i$.

The N=2 superconformal extension of the spacetime conformal transformations,

$$\delta x^{\alpha\dot{\alpha}} = \rho x^{\alpha\dot{\alpha}} + k_{\beta\dot{\beta}}^{\alpha\dot{\alpha}} x^{\beta\dot{\beta}}, \quad (3.10)$$

is dictated by the requirement of preserving the unimodularity and analyticity conditions in eqs. (3.1) and (3.6), respectively. As regards the non-trivial part of the N=2

superconformal transformation laws, one finds [18]

$$\begin{aligned}
\delta x^{\alpha\dot{\alpha}} &= -4i\lambda^{ij}u_i^-u_j^-\theta^{\alpha+}\bar{\theta}^{\dot{\alpha}+} + k_{\beta\dot{\beta}}x^{\alpha\dot{\beta}}x^{\beta\dot{\alpha}} + 4i\left(x^{\alpha\dot{\beta}}\bar{\theta}^{\dot{\alpha}+}\bar{\eta}_{\dot{\beta}}^- - x^{\dot{\alpha}\beta}\theta^{\alpha+}\eta_{\beta}^-\right), \\
\delta\theta^{\alpha+} &= \lambda^{ij}u_i^+u_j^-\theta^{\alpha+} + k_{\beta\dot{\beta}}x^{\alpha\dot{\beta}}\theta^{\beta+} - 2i(\theta^{\beta+}\theta_{\beta}^+)\eta^{\alpha-} + x^{\alpha\dot{\beta}}\bar{\eta}_{\dot{\beta}}^+, \\
\delta\bar{\theta}^{\dot{\alpha}+} &= -\overline{(\delta\theta^{\alpha+})^*}, \\
\delta u_i^+ &= \left[\lambda^{kj}u_k^+u_j^+ + 4ik_{\alpha\dot{\alpha}}\theta^{\alpha+}\bar{\theta}^{\dot{\alpha}+} + 4i\left(\theta^{\alpha+}\eta_{\alpha}^+ + \bar{\eta}_{\dot{\alpha}}^+\bar{\theta}^{\dot{\alpha}+}\right)\right]u_i^-, \\
\delta u_i^- &= 0.
\end{aligned} \tag{3.11}$$

Since the building blocks of invariant actions in HSS are given by the measure, analytic superfields and HSS covariant derivatives, only their transformation properties under the rigid ‘truly’ N=2 superconformal transformations are needed. It follows from eq. (3.11) that [18]

$$\text{Ber}\frac{\partial(\zeta', u')}{\partial(\zeta, u)} = 1 - 2\Lambda, \quad \text{or} \quad \delta[d\zeta^{(-4)}du] = -2\Lambda[d\zeta^{(-4)}du], \tag{3.12}$$

where the HSS superfield parameter

$$\Lambda = -\left(\rho + k_{\alpha\dot{\alpha}}x^{\alpha\dot{\alpha}}\right) + \left(\lambda^{ij} + 4i\theta^{\alpha i}\eta_{\alpha}^j + 4i\bar{\eta}_{\dot{\alpha}}^j\bar{\theta}^{\dot{\alpha} i}\right)u_i^+u_j^- \tag{3.13}$$

has been introduced. Similarly, one easily finds that

$$(D^{++})' = D^{++} - (D^{++}\Lambda)D^0 \quad \text{and} \quad (D^0)' = D^0. \tag{3.14}$$

The truly (rigid) N=2 superconformal infinitesimal parameters can, therefore, be encoded into the single scalar harmonic superfield Λ that is subject to the constraint [18]

$$(D^{++})^2\Lambda = 0, \tag{3.15}$$

and the reality condition

$$\overline{(\Lambda^{++})^*} = \Lambda^{++}, \quad \text{where} \quad \Lambda^{++} \equiv D^{++}\Lambda. \tag{3.16}$$

The transformations rules of the harmonics,

$$\delta u_i^+ = \Lambda^{++}u_i^-, \quad \delta u_i^- = 0, \tag{3.17}$$

together with eqs. (3.12), (3.14), (3.15) and (3.16) yield the very simple and convenient description of rigid N=2 conformal supersymmetry (on the top of N=2 Poincaré supersymmetry) in N=2 HSS.

The *special* hyper-Kähler geometry of the N=2 (rigidly) superconformal NLSM in components was investigated in ref. [19]. A general solution to the special hyper-Kähler geometry in HSS was described in our recent paper [20]. We use the pseudo-real $Sp(1)$ notation for a *Fayet-Sohnius* (FS) hypermultiplet superfield,

$$q_a^+ = (\bar{q}^{*+}, q^+) , \quad a = 1, 2 , \quad q^{a+} = \varepsilon^{ab} q_b^+ , \quad (3.18)$$

which can be easily generalized to the case of several FS hypermultiplets, $q^{a+} \rightarrow q^{A+}$ and $q_A^+ = \Omega_{AB} q^{B+}$, with a constant (antisymmetric) $Sp(k)$ -invariant metric Ω_{AB} , $A, B = 1, \dots, 2k$.

First, we recall that the most general (rigidly) N=2 supersymmetric NLSM can be formulated in terms of the FS hypermultiplet superfields,

$$I_{\text{NLSM}}[q] = -\frac{1}{\kappa^2} \int d\zeta^{(-4)} du \left[\frac{1}{2} q_A^+ D^{++} q^{A+} + \mathcal{K}^{(+4)}(q^{A+}, u_i^\pm) \right] , \quad (3.19)$$

where the real analytic function $\mathcal{K}^{(+4)} = \overline{\mathcal{K}^{(+4)}}^*$ of $U(1)$ charge (+4) is known as a hyper-Kähler (pre-)potential [21].⁵ By manifest N=2 supersymmetry of the NLSM action (3.19), the NLSM metric must be hyper-Kähler for any choice of $\mathcal{K}^{(+4)}$. Unfortunately, an explicit general relation between a hyper-Kähler potential and the corresponding hyper-Kähler metric is not available (see, however, refs. [21, 22, 23] for the explicit hyper-Kähler potentials of (ALF) multi-Taub-NUT and Atiyah-Hitchin metrics, and their derivation from the NLSM (3.19) in HSS, and ref. [24] for a review or a general introduction into the supersymmetric NLSM).

Eq. (3.19) formally solves the hyper-Kähler constraints on the NLSM metric in terms of an arbitrary function $\mathcal{K}^{(+4)}$ that may be considered as the (analytic) hypermultiplet counterpart to the (holomorphic) potential \mathcal{F} of abelian N=2 vector superfields in eq. (2.10). It is, therefore, natural to impose extra N=2 superconformal invariance on the action (3.19), in order to determine a general solution to the special hyper-Kähler geometry, since the free part of the action (3.19) is N=2 superconformally invariant [18]. The FS superfields q^+ have conformal weight one, $\delta q^+ = \Lambda q^+$. Together with eqs. (3.12), (3.17) and (3.19) it implies two constraints [20]:

$$\frac{\partial \mathcal{K}^{(+4)}}{\partial q^{A+}} q^{A+} = 2\mathcal{K}^{(+4)} \quad \text{and} \quad \frac{\partial \mathcal{K}^{(+4)}}{\partial u_i^+} = 0 . \quad (3.20)$$

This means that the special hyper-Kähler potentials are given by *homogeneous* (of degree two) functions $\mathcal{K}^{(+4)}(q^{A+}, u_i^-)$ of q^{A+} . There is no restriction on the dependence of $\mathcal{K}^{(+4)}$ upon u_i^- , though there should be no dependence upon u_i^+ .

⁵The HSS superfields q are dimensionless. The dimensionality of the measure in the action (3.19) is compensated by the coupling constant κ of dimension of length.

The HSS description of the N=2 superconformal *hypermultiplet* actions depending upon the FS superfields in terms of a homogeneous (degree 2) potential is thus formally the same as that of the N=2 superconformal (abelian) vector multiplet actions in the standard N=2 (chiral) superspace (sect. 2). However, the special Kähler geometry in the target space of the NLSM arising in the scalar sector of the N=2 superconformal action (2.10) is very different from the special hyper-Kähler geometry arising from the N=2 superconformal NLSM action (3.19) in components.

4 Ward identities and supergravity compensators

The best way of derivation of the superconformal anomaly relations is based on the use of the *supergravity* (SG) compensators [25]. To compensate the unwanted (local) N=2 superconformal symmetries in N=2 conformal SG, one needs two compensators [26]. This also implies the existence of *two* anomaly relations in the N=2 case (*cf.* sect. 2).

In the case of N=2 superfield SG, its most universal formulation is provided by HSS [18, 27, 28] — see Appendix for a short introduction and our notation. Having an N=2 matter action I coupled to N=2 SG, one can naturally *define* an N=2 supercurrent \mathcal{J} by a variation of the action I with respect to the N=2 conformal SG potential \mathcal{G} ,

$$\mathcal{J} = \frac{\delta I}{\delta \mathcal{G}} , \quad (4.1)$$

in the flat limit where all N=2 conformal SG fields vanish. The N=2 conformal SG potential \mathcal{G} is defined by eq. (A.4), where it is introduced as the general N=2 *harmonic* real superfield, subject to the pre-gauge transformations (A.5) and the gauge transformations (A.7) whose linearized form reads [28]

$$\delta \mathcal{G} = D^{++} l^{--} . \quad (4.2)$$

The N=2 superconformal anomalies are also naturally defined in HSS by variational derivatives of the N=2 matter action with respect to the N=2 SG compensators $v_5^{++}(\zeta, u)$ and $\omega(\zeta, u)$ [28],

$$L^{++} = \frac{\delta I}{\delta v_5^{++}} , \quad (4.3)$$

and

$$\mathcal{T}^{++++} = \frac{\delta I}{\delta \omega} , \quad (4.4)$$

where v_5^{++} is the real *analytic* gauge superfield, associated with the central charge and transforming under the (linearized) gauge transformations (A.13) as

$$\delta v_5^{++} = D^{++} \lambda_5 , \quad (4.5)$$

whereas the *analytic* density ω transforms according to eq. (A.14) that implies (in the linearized approximation)

$$\delta \omega = -D^{--} \Lambda^{++} = -D^{--} (D^+)^4 l^{--} , \quad (4.6)$$

where we have used the last eq. (A.6). The HSS superfields, L^{++} and $\mathcal{T}^{(+4)} \equiv \mathcal{T}^{++++}$, representing the N=2 superconformal anomalies, are analytic by their definition.

Given the N=2 matter action I that is entirely formulated in the ordinary N=2 superspace without harmonics (it is certainly the case in the N=2 gauge sector, without FS hypermultiplets), one can make a connection to the standard N=2 anomaly relation (2.8). The invariance of the action I with respect to the gauge transformations (4.2) obviously yields

$$D^{++} \mathcal{J} = 0 , \quad (4.7)$$

whereas the invariance of the same action with respect to the gauge transformations (4.5) implies

$$D^{++} L^{++} = 0 . \quad (4.8)$$

Equation (4.7) means that \mathcal{J} is independent upon harmonics too, whereas eq. (4.8) implies that $L^{++} = u_i^+ u_j^+ L^{ij}(x, \theta, \bar{\theta})$, where L^{ij} satisfies the N=2 linear multiplet constraints (2.2) due to the analyticity of L^{++} . Finally, the invariance of the action I with respect to the pre-gauge transformations (A.5) and (A.11) yields [28]

$$\frac{i}{4} (D^+)^2 \mathcal{J} = L^{++} , \quad (4.9)$$

which is equivalent to eq. (2.8), as expected.

The HSS results about N=2 SG potentials and compensators imply the natural definitions of the latter in the ordinary N=2 superspace by linear relations,

$$G = \int du \mathcal{G}(\zeta, u, \bar{\theta}^-) \quad (4.10)$$

and

$$\Phi = \int du (\bar{D}^-)^2 v_5^{++}(\zeta, u) . \quad (4.11)$$

The N=2 real superfield $G(x, \theta, \bar{\theta})$ gives the N=2 conformal SG potential (*cf.* ref. [29]), whereas the abelian N=2 (restricted chiral) superfield strength squared, Φ^2 , can serve

as an N=2 (unrestricted) chiral density, i.e. as the N=2 chiral compensator (*cf.* ref. [14]). The anomaly relation (2.8) is then the direct consequence of the definitions

$$J = \frac{\delta I}{\delta G} \quad \text{and} \quad S = \frac{\delta I}{\delta \Phi^2} . \quad (4.12)$$

Having identified the compensator Φ with one of the (abelian) N=2 gauge superfield strengths, $\Phi = W_1$, taking the N=2 gauge LEEA (2.10) to represent the N=2 matter action I above results in the anomalous N=2 superconformal Ward identity (2.12). The second compensator decouples from the effective N=2 gauge matter action, so that it does not have any impact on its anomaly structure. The situation is just the opposite one in the case where the effective N=2 matter action represents a selfinteraction of FS hypermultiplets. The analytic compensator ω can be considered as a part (density) of the hypermultiplet matter, while the invariance of the most general N=2 matter action in HSS with respect to the gauge transformations (4.2) and (4.6) gives rise to the second anomaly relation in the form [28]

$$D^{++} \mathcal{J} = D^{--} \mathcal{T}^{(+4)} . \quad (4.13)$$

Being applied to the hypermultiplet LEEA I in the form of the N=2 NLSM (3.19) in HSS, eq. (4.13) gives rise to the following anomalous N=2 superconformal Ward identity:

$$\sum_A q^{A+} \frac{\partial \mathcal{K}^{(+4)}}{\partial q^{A+}} - 2\mathcal{K}^{(+4)} = \mathcal{T}^{(+4)} . \quad (4.14)$$

This is the key equation in our paper. A generic anomaly $\mathcal{T}^{(+4)}(q, u)$ is analytic, while it has to be invariant under the unbroken gauge symmetries, internal symmetries, and modular transformations, if any. The anomalous Ward identity (4.14) then gives us the equation on the hyper-Kähler potential $\mathcal{K}^{(+4)}$ of the effective N=2 NLSM (LEEA).

5 Examples

To ‘close’ the anomalous N=2 superconformal Ward identity (4.14), as the equation on the effective hyper-Kähler potential $\mathcal{K}(q, u)$, one has to know the anomaly \mathcal{T} as a function of (q, u) explicitly. In the absence of a general solution for the anomaly (at least, we are unaware of it), it is worthy to discuss some examples. The non-anomalous symmetries play the major rôle in determining the form of the anomaly $\mathcal{T}(q, u)$.

The crucial simplification arises when the $SU(2)_R$ automorphisms of N=2 supersymmetry algebra are not broken (together with the N=2 supersymmetry that we

always assume). Since the $SU(2)_R$ transformations are linearly realised in HSS, the R-invariance of the hypermultiplet LEEA amounts to the independence of the anomaly \mathcal{T} (and, hence, of the hyper-Kähler potential \mathcal{K}) upon harmonics. Since both have $U(1)$ charge (+4), the most general (analytic) invariant ‘Ansatz’ is given by a *real quartic polynomial* of the analytic FS superfields q^{A+} [17],

$$\mathcal{T}^{(+4)}(q) \sim \mathcal{K}^{(+4)}(q) = \lambda_{ABCD} q^{A+} q^{B+} q^{C+} q^{D+} , \quad (5.1)$$

whose coefficients $\lambda_{(ABCD)}$ are totally symmetric, being subject to the reality condition, $\overline{\mathcal{K}}^{(+4)} = \mathcal{K}^{(+4)}$. Not all of the coefficients in eq. (5.1) are really significant since the FS kinetic terms in eq. (3.19) have the manifest global $Sp(n)$ symmetry. It may be not accidental that this $Sp(n)$ symmetry coincides with the maximal $Sp(n)$ holonomy group of the hyper-Kähler manifolds in $4n$ real dimensions.

In the case of a single FS hypermultiplet, eq. (5.1) is simplified to

$$\mathcal{T}^{(+4)} \sim \mathcal{K}^{(+4)} = \frac{\lambda}{2} (\overline{q}^+)^2 (q^+)^2 + \left[\gamma \overline{(q^+)}^4 + \beta \overline{(q^+)}^3 q^+ + \text{h.c.} \right] \quad (5.2)$$

with one real (λ) and two complex (β, γ) parameters. The $Sp(1)$ transformations of q_a^+ leave the form of eq. (5.2) invariant, but not its coefficients, which can be used to reduce the number of coupling constants in the family of the hyper-Kähler metrics described by the hyper-Kähler potential (5.2) from five to two. In addition, eq. (5.2) implies the (on-shell) conservation laws

$$D^{++} \mathcal{K}^{(+4)} = D^{++} \mathcal{T}^{(+4)} = 0 , \quad (5.3)$$

which are valid on the equations of motion of the hypermultiplet (FS) superfield,

$$D^{++} \overline{q}^+ = \partial \mathcal{K}^{(+4)} / \partial q^+ \quad \text{and} \quad D^{++} q^+ = -\partial \mathcal{K}^{(+4)} / \partial \overline{q}^+ . \quad (5.4)$$

To understand the physical significance of eq. (5.2), it is instructive to consider first a simpler case, by assuming the additional (translational) $U(1)_T$ symmetry that acts on the complex superfields (q^+, \overline{q}^+) by phase rotations (with a constant parameter α),

$$q^+ \rightarrow e^{i\alpha} q^+ , \quad \overline{q}^+ \rightarrow e^{-i\alpha} \overline{q}^+ , \quad (5.5)$$

but does not move the hyper-Kähler structure in the target space of the N=2 NLSM (3.19). It happens, e.g., in the N=2 supersymmetric QED with a single charged hypermultiplet, or in the Coulomb branch of the Seiberg-Witten model [30]. In geometrical terms, the $U(1)_T$ symmetry amounts to the existence of a tri-holomorphic (translational) isometry in the N=2 NLSM target space. It is worth mentioning

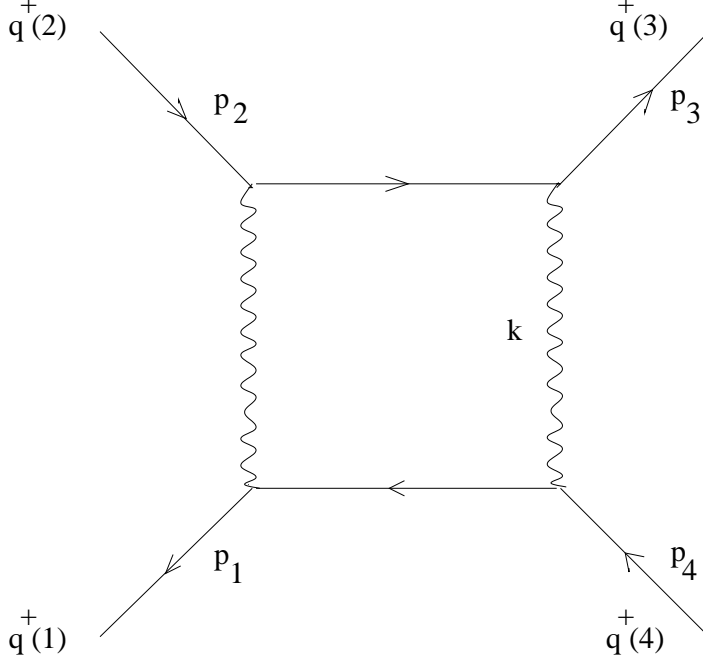


Fig. 1 . The one-loop harmonic supergraph contributing
to the N=2 superconformal anomaly.

that the $SU(2)_R$ isometries are not tri-holomorphic but rotational: they rotate three independent complex structures in the N=2 NLSM hyper-Kähler target space. Given the $SU(2)_R \times U(1)_T$ isometry of the N=2 NLSM target space, it must be the symmetry of the NLSM hyper-Kähler potential, which implies further restrictions in eq. (5.2). The unique, $SU(2)_R \times U(1)_T$ invariant, hyper-Kähler potential is obviously given by

$$\mathcal{K}_{\text{Taub-NUT}}^{(+4)} = \frac{\lambda}{2} \left(\bar{q}^* + q^+ \right)^2, \quad (5.6)$$

while it is known as the hyper-Kähler potential of the Taub-NUT metric with the mass parameter $M = \frac{1}{2}\lambda^{-1/2}$ [22].

The induced coupling constant λ in the one-loop approximation is determined by the HSS graph depicted in Fig. 1. As was shown in ref. [31], this graph does lead to the non-vanishing anomalous contribution having the form (5.6), provided that the matter hypermultiplet has a *non-vanishing* central charge. The wave lines in Fig. 1 denote the analytic propagators of the N=2 (abelian) vector superfields V^{++} (in N=2 supersymmetric Feynman gauge) [32],

$$i \langle V^{++}(1) V^{++}(2) \rangle = \frac{1}{\square_1} (D_1^+)^4 \delta^{12}(\mathcal{Z}_1 - \mathcal{Z}_2) \delta^{(-2,2)}(u_1, u_2), \quad (5.7)$$

where $\delta^{(-2,2)}(u_1, u_2)$ stands for the harmonic delta-function [32]. The hypermultiplet analytic propagators (the solid lines in Fig. 1) with *non-vanishing* central charges are given by [31]

$$i \langle q^+(1) q^+(2) \rangle = \frac{-1}{\square_1 + m^2} \frac{(D_1^+)^4 (D_2^+)^4}{(u_1^+ u_2^+)^3} e^{\tau_3[v(2)-v(1)]} \delta^{12}(\mathcal{Z}_1 - \mathcal{Z}_2) , \quad (5.8)$$

where the ‘bridge’ v satisfies an equation $\mathcal{D}_Z^{++} e^v = 0$, and $m^2 = |Z|^2$ stands for the hypermultiplet (BPS) mass. We find [31]

$$\lambda = \frac{g^4}{\pi^2} \left[\frac{1}{m^2} \ln \left(1 + \frac{m^2}{\Lambda^2} \right) - \frac{1}{\Lambda^2 + m^2} \right] , \quad (5.9)$$

where the gauge coupling constant g and the IR-cutoff Λ have been introduced. It is not difficult to check that $\lambda \neq 0$ only if $Z \neq 0$. The naive ‘non-renormalization theorem’ usually forbids the quantum corrections given by the integrals over a subspace of the full N=2 superspace, like the one in eq. (3.19). However, this ‘theorem’ does not apply here, because of the non-vanishing central charges that give rise to the explicit dependence of the superpropagators (5.8) upon the N=2 superspace Grassmann (anticommuting) coordinates via the bridges $v(\theta, \bar{\theta})$ that are responsible for the N=2 superconformal anomaly.

The more general $SU(2)_R$ -invariant anomaly (5.2) cannot be generated in N=2 perturbation theory, but it can be generated non-perturbatively, due to instanton contributions [33]. Of course, in an abelian N=2 supersymmetric field theory no instantons exist. This means that the N=2 perturbative anomaly described by the Taub-NUT metric is exact in the abelian case. If, however, the underlying N=2 supersymmetric quantum field theory has a non-abelian gauge group of rank larger than one (say, $SU(3)$), then one may expect the nonperturbative contributions to the hypermultiplet LEEA (in the Higgs branch) from instantons and anti-instantons that break the $U(1)_T$ symmetry. The on-shell relations (5.3) imply that in this case both the anomaly $\mathcal{T}^{(+4)}$ and the hyper-Kähler potential $\mathcal{K}^{(+4)}$ can be expressed in terms of a real analytic superfield T^{++++} satisfying the constraints

$$D^{++} T^{++++} = 0 \quad \text{and} \quad \bar{T}^{++++} = T^{++++} , \quad (5.10)$$

which are *the same* as the off-shell constraints (2.6) defining an $O(4)$ projective superfield T^{ijkl} , in the ordinary N=2 superspace, $T^{++++}(\zeta, u) = u_i^+ u_j^+ u_k^+ u_l^+ T^{ijkl}(x, \theta, \bar{\theta})$. Unlike the $O(2)$ tensor multiplet defined by eq. (2.2), the $O(4)$ multiplet does not have a conserved vector amongst its field components. Hence, the $U(1)$ isometry, if any, in the N=2 NLSM to be constructed in terms of T^{++++} , is no longer tri-holomorphic (or translational). The Taub-NUT NLSM (5.6) arises in the limit $T^{++++} \rightarrow (L^{++})^2$.

The two-parametric family of the hyper-Kähler potentials (5.2) describes the (hyper-Kähler and $SU(2)_R$ -invariant) deformations of the *Atiyah-Hitchin* (AH) metric [33]. The N=2 (projective) superspace description of the AH metric in terms of an $O(4)$ projective supermultiplet (2.6) was found in ref. [34]. The AH metric is known to be the only *regular* metric in the family [35]. The ‘difference’ between the AH and Taub-NUT metrics, being considered as the metrics in the quantum moduli space of an N=2 gauge theory (in the region where quantum perturbation theory applies), can be interpreted as the (exponentially small) instanton contributions [36]. Similar remarks are valid in the more general case (5.1) [17].

Another simple example of the N=2 superconformal anomaly, which is still under control, is possible in the case of the unbroken $U(1)_T$ symmetry (5.5). It implies that the function $\mathcal{T}^{(+4)}(q^+, \bar{q}^{*+}; u)$ be a function of the invariant product $(q^+ \bar{q}^{*+})$ and harmonics u_i^- only. The most general ‘Ansatz’ reads

$$\mathcal{T}^{(+4)}(q^+ \bar{q}^{*+}; u) = \sum_{l=0}^{\infty} \xi^{(-2l)} \frac{(\bar{q}^{*+} q^+)^{l+2}}{l+2}, \quad (5.11)$$

whose harmonic-dependent ‘coefficients’ $\xi^{(-2l)}$ are given by

$$\xi^{(-2l)} = \xi^{(i_1 \dots i_{2l})} u_{i_1}^- \dots u_{i_{2l}}^-, \quad l = 1, 2, \dots, \quad (5.12)$$

and obey the reality condition

$$\bar{\xi}^{*(-2l)} = (-1)^l \xi^{(-2l)}. \quad (5.13)$$

The associated solution to eq. (4.14) takes the similar form (5.11), while it appears to be the hyper-Kähler potential describing the multi-Taub-NUT metrics [23]. The multi-Taub-NUT metrics are known to describe static configurations of several (BPS) monopoles [35]. This is not surprising from the viewpoint of the brane engineering of the effective N=2 supersymmetric gauge field theories in M-theory [15], where the N=2 field theory hypermultiplets are associated with (parallel) D6-branes whose configurations in M-theory are just described by the multi-Taub-NUT metrics [37].

The brane technology/M-theory also suggest a possibility of yet another generalization, by adding a (parallel) orientifold $O6^-$ to the D6-branes [37]. In geometrical terms, this means replacing the $U(1)_T$ translational isometry by a rotational $U(1)_R$ isometry in the N=2 NLSM target space. Though the associated N=2 NLSM in HSS were recently constructed in terms of an $O(4)$ projective superfield [20], we are unaware about their explicit reformulation in terms of the FS superfields. We conjecture that this should give rise to the D_k series of the asymptotically locally flat (self-dual) metrics in the four-dimensional target space of N=2 NLSM.

6 Conclusion

In the preceeding section we gave a few non-trivial examples of the $N=2$ superconformal anomaly $\mathcal{T}^{(+4)}$ that saturates the hyper-Kähler potential of the effective hypermultiplet LEEA given by an $N=2$ NLSM. To get more (non-perturbative) solutions to our main eq. (4.14), it may be better to find first the organizing principle behind all those solutions. In the Seiberg-Witten case (2.12), it was the underlying Riemann surface or an integrable system. It is natural to expect a similar hidden curve behind the hypermultiplet LEEA too [17]. The $SU(2)_R$ -invariant effective NLSM metrics in our examples (sect. 5) coincide with the standard metrics in the (BPS) monopole moduli space of the classical $SU(2)$ -Yang-Mills-Higgs system, with magnetic charge $n = 1$ (Taub-NUT) or $n = 2$ (Atiyah-Hitchin). In the $N=2$ NLSM, whose target space metric is given by the monopole moduli space metric of higher magnetic charge $n > 2$, the $SU(2)_R$ symmetry is necessarily broken [17]. As was shown in ref. [38], a BPS monopole of magnetic charge $n > 1$ can always be described by the *spectral curve* (Riemann surface) of genus $(n-1)^2$. It is therefore, conceivable that more general solutions to the anomalous $N=2$ superconformal Ward identity (4.14) are also encoded in terms of the spectral curve, like the Seiberg-Witten-type solutions to eq. (2.12). After a dimensional reduction to three spacetime dimensions, our results support the conjectured mirror symmetry between the Coulomb and Higgs branches [39].

Quantum breaking of conformal symmetry in a (classically) superconformal field theory is related to the appearance of dynamically generated scales. The superconformal compensators can be naturally interpreted as the superfield extensions of the scale parameters. From this physical point of view, the (first) $N=2$ chiral compensator is apparently related to the $N=2$ superfield extension of the renormalization scale squared, whereas the second $N=2$ compensator appears to be the $N=2$ superfield extension of the induced $N=2$ NLSM coupling constant that is (roughly) proportional to the inverse central charge squared (sect. 5). It would be interesting to understand better the physical significance of the second $N=2$ compensator from the viewpoint of the underlying (non-abelian) $N=2$ gauge theory and brane technology.

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Appendix: N=2 supergravity in HSS

All (rigid) N=2 supersymmetric field theories can be naturally defined in HSS, in terms of *unconstrained* analytic superfields, with manifest N=2 supersymmetry [6, 16]. N=2 matter (FS) hypermultiplets are described by complex analytic superfields q^+ of $U(1)$ charge +1, whereas their coupling to N=2 super-Yang-Mills fields is described via the (Lie algebra-valued) extension of the FS hypermultiplet kinetic operator D^{++} by a gauge connection V^{++} , so that the new gauge-covariant operator $\mathcal{D}^{++} = D^{++} + V^{++}$ preserves analyticity. The rigid N=2 superconformal transformations also preserve the analytic subspace of flat N=2 HSS (sect. 3), so it is natural to define N=2 conformal *supergravity* (SG) in HSS along the similar lines [18, 27]: by preserving analyticity (of N=2 matter & gauge fields), the analytic conjugation (= the product of usual complex conjugation and Weyl reflection of the sphere S^2), and unimodularity (of harmonics). In this Appendix we briefly review the HSS formulation of N=2 SG along the lines of ref. [28] — see refs. [16, 18, 27, 28] for more details.

Let $\{\zeta^M, u, \theta^{\hat{\alpha}-}\}$ be the coordinates of the full N=2 HSS in the analytic basis, $\hat{\alpha} = (\alpha, \dot{\alpha})$. The *conformal* N=2 SG transformations are naturally defined in HSS as the analyticity-preserving diffeomorphisms [18, 27],

$$\begin{aligned}\delta\zeta^M &= \lambda^M(\zeta, u) , \\ \delta u^{i+} &= \lambda^{++}(\zeta, u)u^{i-} , \quad \delta u^{i-} = 0 , \\ \delta\theta^{\hat{\alpha}-} &= \lambda^{\hat{\alpha}-}(\zeta, u, \theta^-) .\end{aligned}\tag{A.1}$$

Accordingly, the covariant derivative \mathcal{D}^{++} in N=2 conformal SG can be put into the form

$$\mathcal{D}^{++} = D^{++} + H^{M++}D_M + H^{(+4)}D^{--} + H^{\hat{\alpha}+}D_{\hat{\alpha}}^+ ,\tag{A.2}$$

where the SG *vielbeine* ($H^{M++}, H^{(+4)} \equiv H^{++++}, H^{\hat{\alpha}+}$) have been introduced in front of the flat N=2 HSS covariant derivatives $D_M = (\partial_{\alpha\dot{\alpha}}, D_{\hat{\alpha}}^-)$, D^{--} and $D_{\hat{\alpha}}^+$. The N=2 conformal SG parameters (A.1) can be similarly organized into the one-forms

$$\lambda = \Lambda^M D_M + \Lambda^{++} D^{--} \quad \text{and} \quad \rho = \rho^{\hat{\alpha}-} D_{\hat{\alpha}}^+ .\tag{A.3}$$

Since the SG derivative \mathcal{D}^{++} is supposed to preserve analyticity, $D_{\hat{\alpha}}^+ \mathcal{D}^{++} \Phi^{(+p)} = 0$, of the analytic superfield $\Phi^{(+p)}$ of a positive $U(1)$ charge p , $D_{\hat{\alpha}}^+ \Phi^{(+p)} = 0$, the vielbeine of eq. (A.2) have to obey certain linear constraints, whose solution is given by [28]

$$H^{\alpha\dot{\alpha}++} = -iD^{\alpha+}\bar{D}^{\dot{\alpha}+}\mathcal{G} , \quad H^{\alpha+++} = -\frac{1}{8}D^{\alpha+}(\bar{D}^+)^2\mathcal{G} , \quad H^{(+4)} = (D^+)^4\mathcal{G} ,\tag{A.4}$$

where $(D^+)^4 = \frac{1}{16}(D^+)^2(\bar{D}^+)^2$, and the real unconstrained (general HSS superfield) N=2 SG pre-potential $\mathcal{G}(\zeta, u, \theta^-)$ is subject to the pre-gauge transformations [28]

$$\delta\mathcal{G} = \frac{1}{4}(D^+)^2\Omega^{--} + \frac{1}{4}(\bar{D}^+)^2\bar{\Omega}^{*-} \quad (A.5)$$

with the complex unconstrained HSS parameter $\Omega^{--}(\zeta, u, \theta^-)$. Accordingly, the gauge HSS superfield parameters $(\Lambda^M, \Lambda^{++})$ are to be expressed in terms of a single real unconstrained HSS superfield $l^{--}(\zeta, u, \theta^-)$ [28],

$$\Lambda^{\alpha\dot{\alpha}} = -iD^{\alpha+}\bar{D}^{\dot{\alpha}+}l^{--}, \quad \Lambda^{\alpha+} = -\frac{1}{8}D^{\alpha+}(\bar{D}^+)^2l^{--}, \quad \Lambda^{++} = (D^+)^4l^{--}. \quad (A.6)$$

It is easy to see that $H^{\hat{\alpha}+}$ is pure gauge, so that we can simply ignore it, $H^{\hat{\alpha}+} = 0$. Then the transformation law of \mathcal{G} under the remaining gauge transformations with the parameters (λ, l^{--}) is given by a simple formula,

$$\delta\mathcal{G} = -\lambda\mathcal{G} + \mathcal{D}^{++}l^{--}. \quad (A.7)$$

In the absence of superconformal anomalies, there exist a (WZ-type) gauge, where the HSS superfield \mathcal{G} is independent upon harmonics, being subject to the constraints

$$D_{ij}\mathcal{G}(x, \theta, \bar{\theta}) = \bar{D}_{ij}\mathcal{G}(x, \theta, \bar{\theta}) = 0. \quad (A.8)$$

This equation coincides with eq. (2.7) defining the N=2 conformal supercurrent, so that the independent field components of \mathcal{G} (in the WZ gauge) are in one-to-one correspondence with the field content of the off-shell N=2 conformal SG, as it should.

To construct N=2 matter and gauge couplings in N=2 Poincaré (non-conformal or Einstein) SG, one has to *compensate* ‘truly’ N=2 superconformal gauge transformations, namely, dilatations, special conformal transformations, $U(1)$ chiral and $SU(2)_{\text{conf}}$ rotations, and N=2 special supersymmetry [26]. Some of them are compensated by a real analytic gauge superfield $V_5^{++}(\zeta, u)$ as the *first compensator*, subject to abelian gauge transformations in HSS,

$$\delta V_5^{++} = \mathcal{D}^{++}\Lambda_5, \quad (A.9)$$

with the real analytic parameter $\Lambda_5(\zeta, u)$. In the context of gauging the N=2 supersymmetry algebra, the V_5^{++} superfield is associated with the central charge generator \hat{Z} . Though the N=2 conformal SG itself has the vanishing central charge, N=2 matter hypermultiplets may have non-vanishing central charges. It is natural to incorporate the central charge generator into the HSS covariant derivatives by redefining \mathcal{D}^{++} to $\mathcal{D}_Z^{++} = \mathcal{D}^{++} + V_5^{++}\hat{Z}$, etc. At the component level the N=2 supermultiplet V_5^{++} (in a WZ gauge) adds extra $8_B + 8_F$ off-shell (field) degrees of freedom to the N=2 Weyl

multiplet, thus forming together the so-called *minimal* off-shell N=2 SG multiplet with $32_B + 32_F$ field components [26].

The extended HSS covariant derivative \mathcal{D}_Z^{++} also has to preserve analyticity, which implies a linear constraint on V_5^{++} . A solution to the constraint reads [28]

$$V_5^{++} = \frac{i}{4}(D^+)^2\mathcal{G} - \frac{i}{4}(\bar{D}^+)^2\mathcal{G} + v_5^{++} , \quad (\text{A.10})$$

where v_5^{++} is real analytic. The pre-gauge transformations (A.5) are to be appended by [28]

$$\delta v_5^{++} = i(D^+)^4(\Omega^{--} - \bar{\Omega}^{--*}) . \quad (\text{A.11})$$

The related restrictions on the HSS parameter Λ_5 in eq. (A.9) are [28]

$$\Lambda_5 = \frac{i}{4}(D^+)^2 l^{--} - \frac{i}{4}(\bar{D}^+)^2 l^{--} + \lambda_5 , \quad (\text{A.12})$$

where λ_5 is real analytic. One easily finds

$$\delta v_5^{++} = -\lambda v_5^{++} + \mathcal{D}^{++}\lambda_5 . \quad (\text{A.13})$$

To compensate the remaining unwanted gauge symmetries (e.g., $SU(2)_{\text{conf.}}$), one needs a *second compensator* that may have either a finite or the infinite number of the auxiliary field components. The three *minimal* formulations of N=2 Poincaré SG, each having $40_B + 40_F$ off-shell (field) degrees of freedom, were described in ref. [26]. Unfortunately, all of them impose some restrictions on allowed N=2 matter couplings, which makes them of limited use in the context of generic N=2 NLSM. The most universal choice is given by a *real analytic* compensator ω that compensates the N=2 local supersymmetry transformations of the analytic measure,

$$\omega'(\zeta', u') = Ber^{-1} \left(\frac{\partial(\zeta', u')}{\partial(\zeta, u)} \right) \omega(\zeta, u) . \quad (\text{A.14})$$

It allows one to accommodate any N=2 NLSM in N=2 SG via the ‘covariantization’ of the (rigid) N=2 NLSM action in HSS, by using the invariant analytic measure $d\zeta^{(-4)}du\omega$. It is the absence of an analytic density that is responsible for the restricted N=2 matter couplings in the more conventional formulations of N=2 SG [26]. Coupling to N=2 SG deforms hyper-Kähler geometry of the N=2 NLSM target space into quaternionic geometry [40, 41]. The actions of quaternionic NLSM coupled to N=2 SG in HSS were recently investigated in ref. [42], where the density ω was constructed in terms of a FS hypermultiplet superfield q^+ as $\omega \sim (u_a^- q^{a+})^2$. When going in the opposite direction, a rigidly N=2 superconformal (special hyper-Kähler) NLSM arises from the quaternionic one after putting all the N=2 conformal SG fields to zero, $\mathcal{G} = 0$, together with the vanishing Maxwell multiplet, $v_5^{++} = 0$, and $\omega = 1$.

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